2. Dot Product and Cross Product

In this lecture, we will discuss

- The Dot Product
  - Definition and Properties
  - Geometric interpretation
  - Test for orthogonality of vectors
  - Angle between vectors
  - Orthonormal set of vectors
  - Vector expressed in terms of orthogonal vectors
- The Cross Product
  - Definition and Properties
  - Geometric interpretation
  - Area of the parallelogram spanned by two vectors
  - Volume of the parallelepiped spanned by three vectors

The Dot Product

Definition. Dot Product
Let $\mathbf{v} = (v_1, \ldots, v_n)$ and $\boldsymbol{w} = (w_1, \ldots, w_n)$ be vectors in $\mathbb{R}^n, n \geq 2$. Then

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + \cdots + v_n w_n. \in \mathbb{R}$$

In particular, if $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$, then

$$\mathbf{v} \cdot \mathbf{w} = (v_1 \mathbf{i} + v_2 \mathbf{j}) \cdot (w_1 \mathbf{i} + w_2 \mathbf{j}) = v_1 w_1 + v_2 w_2,$$

and

$$\mathbf{v} \cdot \mathbf{w} = (v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}) \cdot (w_1 \mathbf{i} + w_2 \mathbf{j} + w_3 \mathbf{k}) = v_1 w_1 + v_2 w_2 + v_3 w_3$$

if $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^3$.

Theorem 1. Properties of the Dot Product

Assume that $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ are vectors in $\mathbb{R}^n$ (for $n \geq 2$), and α is a real number. Then

- $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$ (commutative)
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ (distributive with respect to addition)
- $(\alpha \mathbf{u}) \cdot \mathbf{v} = \alpha(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (\alpha \mathbf{v})$ (distributive with respect to scalar multiplication)
- $\mathbf{0} \cdot \mathbf{v} = 0$ ($\mathbf{0}$ is the zero vector)
- $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$.
- If $\mathbf{v}$ and $\mathbf{w}$ are parallel, then $\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|$ if $\mathbf{v}$ and $\mathbf{w}$ have the same direction,
- and $\mathbf{v} \cdot \mathbf{w} = -\|\mathbf{v}\|\|\mathbf{w}\|$ if they have opposite directions.

Theorem 2. Geometric Version of the Dot Product

Let $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$. Then

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \cos\theta,$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

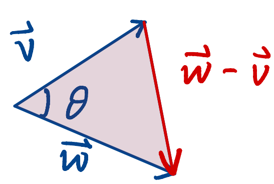
Outline of the proof:

- If $\vec{v}$ or $\vec{w}$ is $\vec{0}$, then both sides of the eqn are 0
- If $\vec{v}$ and $\vec{w}$ are parallel ($\theta = 0$, or π), it's easy to show

  $\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cos\theta$

  1 or −1. if

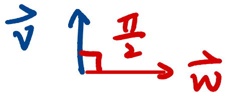
- If $\vec{v} \neq \vec{0}$, $\vec{w} \neq \vec{0}$, and $0 < \theta < \pi$.



  Law of cosine: $\|\vec{v} - \vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\| \cdot \|\vec{w}\| \cos\theta$

  Property of dot product: $\|\vec{v} - \vec{w}\|^2 = (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = \|\vec{v}\|^2 - 2\vec{v} \cdot \vec{w} + \|\vec{w}\|^2$

Compare the RHS of the equations, we get $\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cos\theta$.



$\cos\frac{\pi}{2} = 0$

Theorem 3. Test for Orthogonality of Vectors
Let $\mathbf{v}$ and $\mathbf{w}$ be nonzero vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$. Then $\mathbf{v}\cdot\mathbf{w} = 0$ if and only if $\mathbf{v}$ and $\mathbf{w}$ are orthogonal.

Definition. Orthonormal Set of Vectors
Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ (where $k \geq 2$) in $\mathbb{R}^n, n \geq 2$ are said to form an orthonormal set if they are of unit length and each vector in the set is orthogonal to the others.

Theorem 4. Angle Between Vectors
Let $\mathbf{v}$ and $\mathbf{w}$ be nonzero vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$. Then

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|},$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

Example 1. Find the angle θ between the vectors $\mathbf{v} = (2, 1, -1)$ and $\mathbf{w} = (3, -4, 1)$.

ANS: Since $\vec{v}\cdot\vec{w} = 2\times 3 - 1\times 4 - 1\times 1 = 1$

$$\|\vec{v}\| = \sqrt{2^2+1^2+1^2} = \sqrt{6}$$

$$\|\vec{w}\| = \sqrt{3^2+4^2+1^2} = \sqrt{26}$$

then

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\cdot\|\vec{w}\|} = \frac{1}{\sqrt{6}\cdot\sqrt{26}} = \frac{1}{2\sqrt{29}}$$

$$\approx 0.08$$

$$\Rightarrow \theta \approx 1.491 \text{ rad}$$

Theorem 5. Vector Expressed in Terms of Orthogonal Vectors
Let $\mathbf{v}$ and $\mathbf{w}$ be (nonzero) orthogonal vectors in $\mathbb{R}^2$ and let $\mathbf{a}$ be any vector in $\mathbb{R}^2$. Then

$$\mathbf{a} = a_{\mathbf{v}}\mathbf{v} + a_{\mathbf{w}}\mathbf{w},$$

where $a_{\mathbf{v}} = \dfrac{\mathbf{a}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}$ is the component of $\mathbf{a}$ in the direction of $\mathbf{v}$ and $a_{\mathbf{w}} = \dfrac{\mathbf{a}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}$ is the component of $\mathbf{a}$ in the direction of $\mathbf{w}$ (or in the direction orthogonal to $\mathbf{v}$).

Special case : Let $\vec{v} = \vec{i} = (1,0)$, $\vec{w} = \vec{j} = (0,1)$

Then $\vec{a} = a_1\vec{i} + a_2\vec{j}$. if $\vec{a} = (a_1, a_2)$

when $a_1 = \vec{a}\cdot\vec{i}$, $a_2 = \vec{a}\cdot\vec{j}$

"Dot products give the value of the coordinates"

Proof: From Linear algebra, we know $\vec{a}$ can be written as a linear combination of two mutually orthogonal vectors.

$$\vec{a} = a_{\vec{v}}\vec{v} + a_{\vec{w}}\vec{w} \text{ for some } a_{\vec{v}}, a_{\vec{w}} \in \mathbb{R}.$$

Take the dot product of $\vec{a} = a_{\vec{v}}\vec{v} + a_{\vec{w}}\vec{w}$ with $\vec{v}$, we have

$$\vec{a}\cdot\vec{v} = a_{\vec{v}}\,\vec{v}\cdot\vec{v} + a_{\vec{w}}\,\cancel{\vec{w}\cdot\vec{v}} \quad 0$$

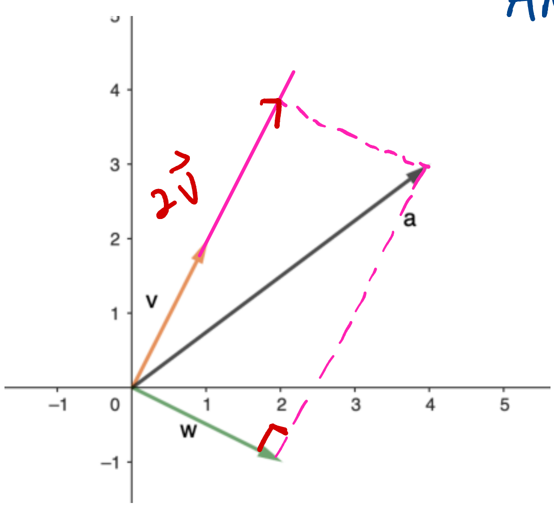
Thus

$$a_{\vec{v}} = \frac{\vec{a}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}$$

Similarly,

$$a_{\vec{w}} = \frac{\vec{a}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}$$

Example 2. Check that $\mathbf{v} = (1, 2)$, and $\mathbf{w} = (2, -1)$ are mutually orthogonal vectors and express $\mathbf{a} = (4, 3)$ in terms of $\mathbf{v}$, and $\mathbf{w}$.



ANS: Since $\vec{v} \cdot \vec{w} = 1 \times 2 - 2 \times 1 = 0$

$\vec{v} \perp \vec{w}$

By the above theorem

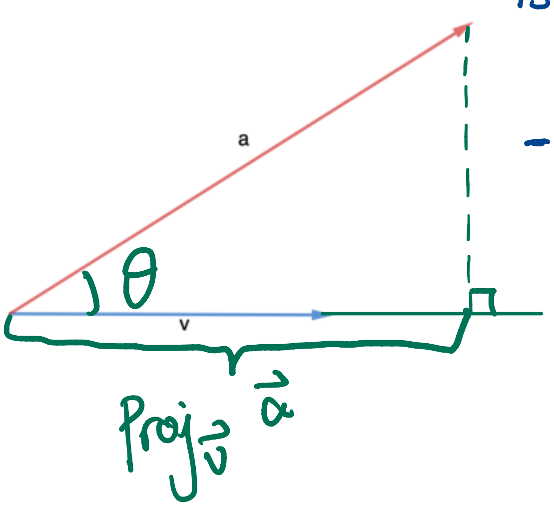
$$\vec{a} = a_{\vec{v}} \vec{v} + a_{\vec{w}} \vec{w}$$

where $a_{\vec{v}} = \dfrac{\vec{a} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} = \dfrac{1 \times 4 + 2 \times 3}{5} = 2$

$$a_{\vec{w}} = \frac{\vec{a} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} = \frac{2 \times 4 - 3 \times 1}{5} = 1$$

Thus $\vec{a} = 2\vec{v} + \vec{w}$

Projection of $\mathbf{a}$ onto $\mathbf{v}$.



In fact, $a_{\vec{v}} \vec{v}$ in the Eq $\vec{a} = a_{\vec{v}} \vec{v} + a_{\vec{w}} \vec{w}$ is the vector projection of $\vec{a}$ onto $\vec{v}$ ($\mathrm{proj}_{\vec{v}} \vec{a}$)

- The scalar projection.

$$\|\mathrm{proj}_{\vec{v}} \vec{a}\| = \|\vec{a}\| \cdot \cos\theta$$

$$= \cancel{\|\vec{a}\|} \cdot \frac{\vec{a} \cdot \vec{v}}{\cancel{\|\vec{a}\|} \cdot \|\vec{v}\|}$$

$$= \frac{\vec{a} \cdot \vec{v}}{\|\vec{v}\|}$$

- The projection vector of $\vec{a}$ onto $\vec{v}$

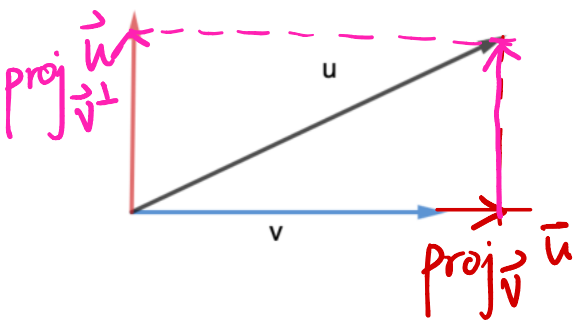
$$\mathrm{proj}_{\vec{v}} \vec{a} = \underbrace{\|\mathrm{proj}_{\vec{v}} \vec{a}\|}_{\text{length}} \cdot \underbrace{\frac{\vec{v}}{\|\vec{v}\|}}_{\text{unit vector}} = \frac{\vec{a} \cdot \vec{v}}{\|\vec{v}\|} \cdot \frac{\vec{v}}{\|\vec{v}\|} = \frac{\vec{a} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v}$$

$$= a_{\vec{v}} \vec{v}$$

Example 3. Let $\mathbf{u} = (-2, 3, -1)$ and $\mathbf{v} = (-1, 1, 1)$. Compute

(1) the projection of $\mathbf{u}$ along $\mathbf{v}$, and

(2) the projection of $\mathbf{u}$ orthogonal to $\mathbf{v}$.



ANS: (1) By the previous discussion, we know

$$\text{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v}$$

$$= \frac{2+3-1}{1+1+1} (-1, 1, 1)$$

$$= \frac{4}{3} (-1, 1, 1),$$

(2) Let $\vec{v}^{\perp}$ denote the vector orthogonal to $\vec{v}$ (with the same length), Then it's not hard to check

$$\text{Proj}_{\vec{v}^{\perp}} \vec{u} = \vec{u} - \text{proj}_{\vec{v}} \vec{u} = (-2, 3, 1) - \frac{4}{3} (-1, 1, 1)$$

$$= \frac{1}{3} (-2, 5, -7)$$

The Cross Product

Definition Cross Product

The cross product of two vectors $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$ is the vector $\mathbf{c} = \mathbf{v} \times \mathbf{w}$ in $\mathbb{R}^3$ defined by

$$\mathbf{c} = \mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

$$= (v_2w_3 - v_3w_2)\mathbf{i} - (v_1w_3 - v_3w_1)\mathbf{j} + (v_1w_2 - v_2w_1)\mathbf{k}$$

Example 4. Compute $\mathbf{v} \times \mathbf{w}$ and $\mathbf{w} \times \mathbf{v}$, if $\mathbf{v} = \mathbf{i} - 2\mathbf{k}$ and $\mathbf{w} = -2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$.

ANS:

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2 \\ -2 & 3 & -4 \end{vmatrix} = \vec{i}\begin{vmatrix} 0 & -2 \\ 3 & -4 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & -2 \\ -2 & -4 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 0 \\ -2 & 3 \end{vmatrix}$$

$$= 6\vec{i} + 8\vec{j} + 3k$$

Similarly.

$$\vec{w} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 3 & -4 \\ 1 & 0 & -2 \end{vmatrix} = -6\vec{i} - 8\vec{j} - 3\vec{k}$$

Note $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$. In general, it's true.

Theorem 6. Properties of the Cross Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, be vectors in $\mathbb{R}^3$ and let α be any real number. The cross product satisfies

- $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$ (anticommutativity),
- $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
- $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$ (distributivity with respect to the sum).
- $\mathbf{v} \times \mathbf{v} = \mathbf{0}$ ($\mathbf{0}$ is the zero vector in $\mathbb{R}^3$)
- $\alpha(\mathbf{v} \times \mathbf{w}) = (\alpha\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (\alpha\mathbf{w})$.

By the definitions of dot product and cross product, we have

Lemma 1. Let $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, and $\mathbf{w} = (w_1, w_2, w_3)$, then

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

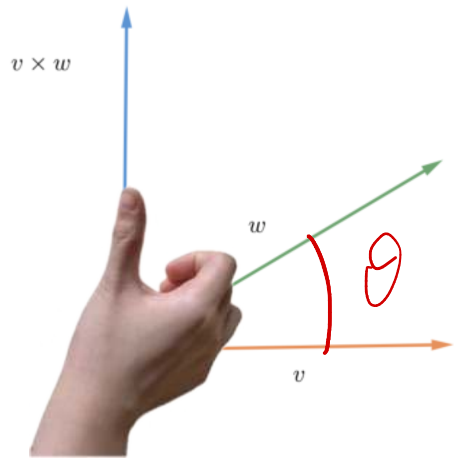
Let A be a matrix with rows formed by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. By **Lemma 1**, we know $\det(A) = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

Theorem 7. Geometric Properties of the Cross Product

Let $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^3$. Then
(a) The cross product $(\mathbf{v} \times \mathbf{w})$ is a vector orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.
(b) The magnitude of $\mathbf{v} \times \mathbf{w}$ is given by $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| \sin\theta$, where θ denotes the angle between $\mathbf{v}$ and $\mathbf{w}$.

Right-hand rule

- Place your right hand in the direction of $\mathbf{v}$, and curl your fingers from $\mathbf{v}$ to $\mathbf{w}$ through the angle θ (remember that θ is the smaller of the two angles formed by the lines with directions $\mathbf{v}$ and $\mathbf{w}$).
- Your thumb then points in the direction of $\mathbf{v} \times \mathbf{w}$.



Theorem 8. Test for Parallel Vectors

Nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$ are parallel if and only if $\mathbf{v} \times \mathbf{w} = \mathbf{0}$.

🤔 **Question.** Let $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^3$ and θ be the angle between them, can you express $\tan\theta$ using the dot and cross products of $\mathbf{v}$ and $\mathbf{w}$?

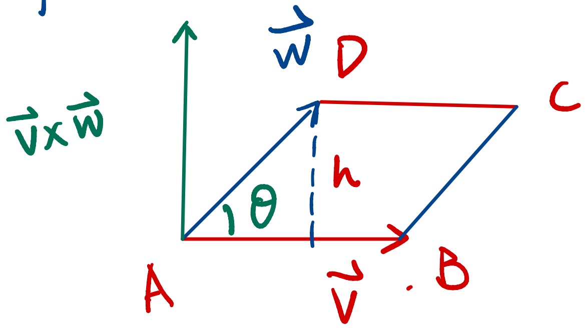
ANS: Since $\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos\theta$

$$\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \sin\theta$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\dfrac{\|\vec{v} \times \vec{w}\|}{\cancel{\|\vec{v}\| \cdot \|\vec{w}\|}}}{\dfrac{\vec{v} \cdot \vec{w}}{\cancel{\|\vec{v}\| \cdot \|\vec{w}\|}}} = \frac{\|\vec{v} \times \vec{w}\|}{\vec{v} \cdot \vec{w}}$$

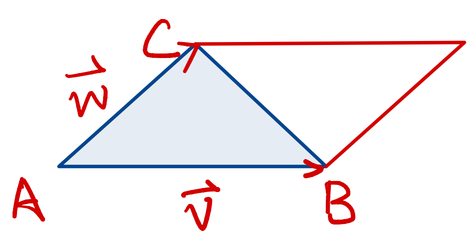
Theorem 9. Area of the Parallelogram Spanned by Two Vectors
Let $\mathbf{v}$ and $\mathbf{w}$ be nonzero, nonparallel vectors in $\mathbb{R}^3$. The magnitude $\|\mathbf{v}\times\mathbf{w}\|$ is the real number equal to the area of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$.

Proof.



$$\text{Area}_{ABCD} = \|\vec{v}\|\cdot h$$
$$= \|\vec{v}\|\cdot\|\vec{w}\|\cdot\sin\theta$$
$$= \|\vec{v}\times\vec{w}\|$$
by Thm 7(b)

Exercise 5. Find the area of the triangle with vertices $(0,2,1)$, $(3,3,3)$, and $(-1,4,2)$.



ANS. Computing the area of the parallelogram with vertices located at $A(0,2,1)$, $B(3,3,3)$ and $C(-1,4,2)$ and then dividing by 2 will yield the area of the triangle in question.

Let $\mathbf{v}$ and $\mathbf{w}$ be the vectors determined by the directed line segments $\overrightarrow{AB}$ and $\overrightarrow{AC}$ respectively.

Then $\mathbf{v}=(3,1,2)$ and $\mathbf{w}=(-1,2,1)$ and hence

$$\mathbf{v}\times\mathbf{w}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & 1 & 2\\ -1 & 2 & 1\end{vmatrix}=(1-4,-3-2,6+1)=(-3,-5,7).$$

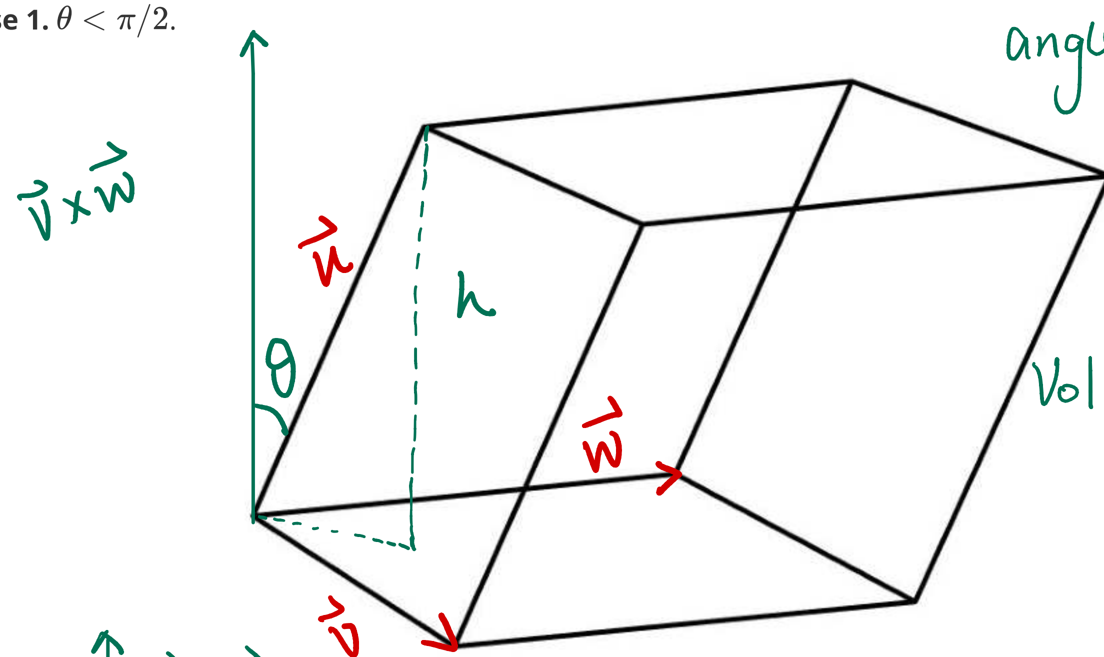
Therefore, the area of the triangle is $\|\mathbf{v}\times\mathbf{w}\|/2=\|(-3,-5,7)\|/2=\sqrt{83}/2$.

Volume of the parallelepiped spanned by three vectors

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be nonzero vectors in $\mathbb{R}^3$ such that $\mathbf{v}$ and $\mathbf{w}$ are not parallel (so that they span a parallelogram) and such that $\mathbf{u}$ does not belong to the plane spanned by $\mathbf{v}$ and $\mathbf{w}$.

Construct the parallelepiped spanned by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in the following figure.

Case 1. $\theta < \pi/2$.

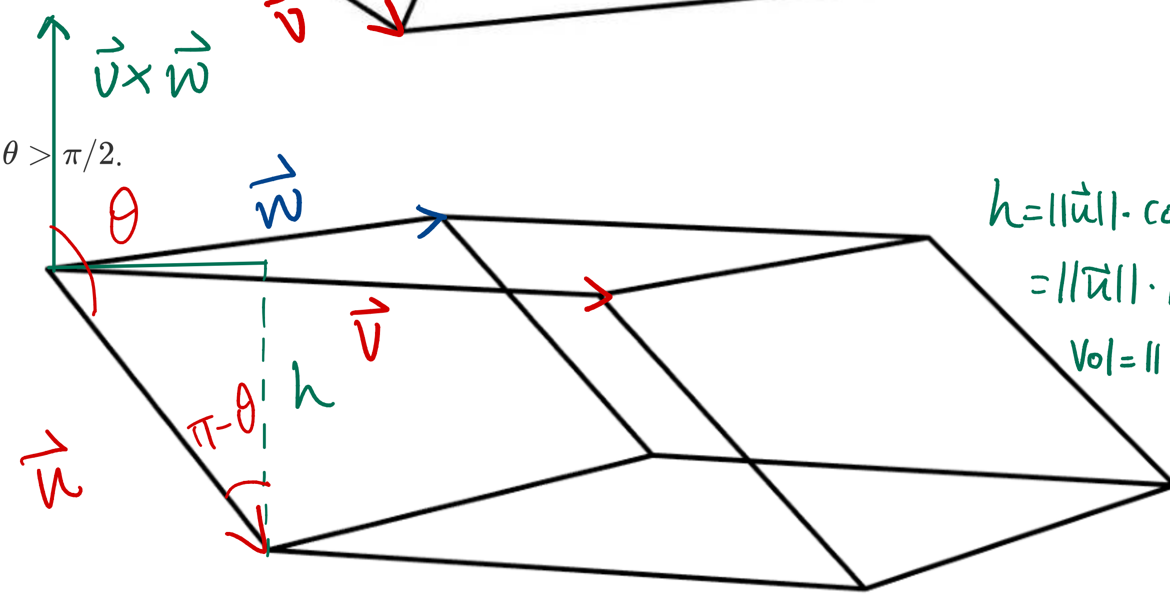


Let θ be the angle between $\vec{u}$ and $\vec{v} \times \vec{w}$

$h = \|\vec{u}\| \cos\theta$

$\text{Vol} = \|\vec{v} \times \vec{w}\| \cdot \|\vec{u}\| \cdot \cos\theta$

Case 2. $\theta > \pi/2$.



$h = \|\vec{u}\| \cdot \cos(\pi - \theta)$

$= \|\vec{u}\| \cdot |\cos\theta|$

$\text{Vol} = \|\vec{v} \times \vec{w}\| \cdot \|\vec{u}\| \cdot |\cos\theta|$

- $\|\mathbf{v} \times \mathbf{w}\|$ is the area of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$.
- If $\theta < \pi/2$, the height h of the parallelepiped is $h = \|\mathbf{u}\| \cos\theta$.
- If $\theta > \pi/2$, then $h = \|\mathbf{u}\| \cos(\pi - \theta) = -\|\mathbf{u}\| \cos\theta$. In either case, $h = \|\mathbf{u}\| |\cos\theta|$.
- Therefore,

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = \|\mathbf{v} \times \mathbf{w}\| \|\mathbf{u}\| |\cos\theta|$$

is the volume of the parallelepiped spanned by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

- Let A be the matrix with rows as $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. By **Lemma 1**, we know

$$\det(A) = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

- Thus, we have $|\det(A)| = \mathrm{vol}(P)$.
- This is often refered as "the absolute value of the determinate gives the value of the volumn".

Remark. Volume and Determinate

- The notion of parallelepiped can be generalized in $\mathbb{R}^n$ and so does the notion of the volume of the parallelepiped. The equation $|\det(A)| = \mathrm{vol}(P)$ still holds once those concepts are properly generalized.
- The proof of this is not trivial. You can refer to this webpage if you are curious about it.